1. $(1-i)^{1+i}$

$$= e^{(1+i)\log(1-i)}$$

$$= e^{(1+i)\log(\sqrt{2}e^{-\frac{\pi}{4}i})}$$

$$= e^{(1+i)(\frac{1}{2}\ln 2 - \frac{\pi}{4}i)}$$

$$= e^{\frac{1}{2}\ln 2 + \frac{\pi}{4}} e^{(\frac{1}{2}\ln 2 - \frac{\pi}{4})i}$$

$$= \sqrt{2}e^{\frac{\pi}{4}}\cos(\tfrac{1}{2}\ln 2 - \tfrac{\pi}{4}) + \sqrt{2}e^{\frac{\pi}{4}}\sin(\tfrac{1}{2}\ln 2 - \tfrac{\pi}{4})i$$

$\square$

2 (i) $\cos z = \dfrac{e^{iz} + e^{-iz}}{2}$

$$= \frac{e^{i(x+iy)} + e^{-i(x+iy)}}{2}$$

$$= \frac{e^{-y}e^{ix} + e^{y}e^{-ix}}{2}$$

$$= \frac{2e^{ix}e^{-y} + 2e^{-ix}e^{y}}{4}$$

$$= \frac{(e^{ix}+e^{-ix})(e^{y}+e^{-y})}{4} - \frac{(e^{ix}-e^{-ix})(e^{y}-e^{-y})}{4}$$

$$= \cos x\cosh y - i\sin x\sinh y$$

or $\cos z = \cos(x+iy)$

$$= \cos x\cos(iy) - \sin x\sin(iy)$$

$$= \cos x\cosh(y) - i\sin x\sinh y.$$

$\square$

(ii) Let $u(x,y) = \cos x \cosh y$

$v(x,y) = -\sin x \sinh y$.

By (i), $\cos z = u(x,y) + i v(x,y)$.

$$\frac{\partial}{\partial x} u(x,y) = -\sin x \cosh y \qquad \frac{\partial}{\partial y} u(x,y) = \cos x \sinh y$$

$$\frac{\partial}{\partial x} v(x,y) = -\cos x \sinh y \qquad \frac{\partial}{\partial y} v(x,y) = -\sin x \cosh y$$

Then $\begin{cases} \frac{\partial}{\partial x} u = \frac{\partial}{\partial y} v \\ \frac{\partial}{\partial y} u = -\frac{\partial}{\partial x} v \end{cases}$ at all points on $\mathbb{C}$.

By Cauchy-Riemann equation, $\cos z$ is differentiable at all points on $\mathbb{C}$.

$\square$

3. When z_0 is inside γ, i.e., $|z_0| < 1$,

$$\int_\gamma \frac{z^4 + z^2 + 1}{(z - z_0)^4} dz = \frac{2\pi i}{3!} \left(\frac{d^3}{dz^3} (z^4 + z^2 + 1) \right) \Bigg|_{z = z_0}$$

$$= \left(\frac{\pi i}{3}\right) \cdot (24 z_0)$$

$$= 8 \pi i z_0$$

- When z_0 is outside γ, i.e., $|z_0| > 1$,

$$\frac{z^4+z^2+1}{(z-z_0)^4} \text{ is analytic inside and on } \gamma.$$

By Cauchy-Goursat Theorem, $\displaystyle\int_\gamma \frac{z^4+z^2+1}{(z-z_0)^4}\,dz = 0$.

$\square$

4 (Method 1)

Let $g(z) = u_x - iu_y$

Then $(u_x)_x = u_{xx}$ $\quad (-u_y)_x = -u_{yx}$

$(u_x)_y = u_{xy}$ $\quad (-u_y)_y = -u_{yy}$

Since u is harmonic, (of course u is twice continuously differentiable), then

$u_{xx} = -u_{yy}$, $\quad u_{xy} = -(-u_{yx})$

By Cauchy-Riemann equation, g is analytic

Fix any $z_0 \in \Omega$, define $f = \int_{z_0}^{z} g(s)\,ds + u(z_0)$

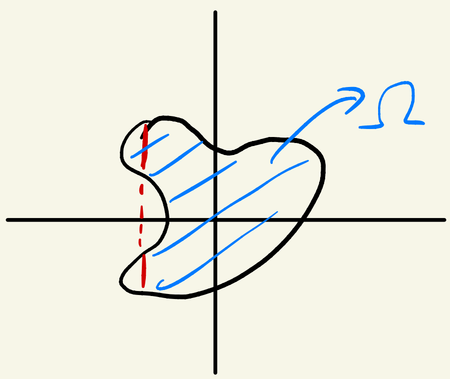
Since Ω is simply connected, the integral is independent of path. Then f is analytic and

$\operatorname{Re}(f) = u$

(Remark: I realized that simple connectedness is not sufficient to define $\int_{y_0}^{y} u_x(x,t)\,dy$

We need convexity of Ω.

Even if Ω is simply connected, we could also have some problem, e.g.



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